ELSEVIER

Journal of Geometry and Physics 35 (2000) 93–98

JOURNAL OF GEOMETRY AND PHYSICS

Brief communication

A note on real Killing spinors in Weyl geometry ☆

Volker Buchholz

Humboldt Universität zu Berlin, Institute für Reine Mathematik, Rudower Chaussee 25, D-12489 Berlin, Germany

Received 12 January 2000

Abstract

This note is dedicated to the real Killing equation on three-dimensional Weyl manifolds. Any manifold admitting a real Killing spinor of weight 0 satisfies the conditions of a Gauduchon–Tod geometry. Conversely, any simply connected Gauduchon–Tod geometry has a two-dimensional space of solutions of the real Killing equation on the spinor bundle of weight 0. © 2000 Elsevier Science B.V. All rights reserved.

MSC: 53C05; 53C10; 53A30

Subj. Class: Differential geometry

Keywords: Weyl geometry; Spin geometry; Killing equation

1. Introduction

In [2], we gave an introduction to spinor geometry on Weyl manifolds and investigated the Dirac-, Twistor-and Killing equation in this context. Concerning the real Killing equation we presented in [2] the following result:

Theorem 1.1 (see [2], Theorem 3.1). *Let $\psi \in \Gamma(S^w)$ be a real Killing spinor on a Weyl manifold (M^n, c, W), i.e. there exists a density $\beta \in \Gamma(\mathcal{L}^{-1})$ for which*

$$\nabla^{S,w}\phi = \beta \otimes \nu\psi,$$

is satisfied. Then the following statements hold:
1. $R = 4n(n-1)\beta^2$.

☆ Supported by the SFB 288 of the DFG.
E-mail address: bv@mathematik.hu-berlin.de (V. Buchholz)

PII: S0393-0440(00)00014-0

2. $w \neq 0$: *W is exact and Einstein–Weyl.*
3. $w = 0, n \geq 4$: *W is exact and Einstein–Weyl.*

The following equations were obtained within the proof and will play an important role in the sequel:

$$\mu^2 Ric' \otimes \psi = 2\left(n-1-\frac{n-1}{n-2}\right)\nabla\beta\otimes\psi + \left(1-\frac{n-1}{n-2}\right)\mu^{12} Alt\,\nabla\beta\otimes c\otimes\psi + \frac{R}{n}\nu\psi - \mu^2 F\otimes\psi, \tag{1}$$

$$F\cdot\psi = -\frac{4(n-1)}{n-2}\nabla\beta\cdot\psi. \tag{2}$$

Theorem 1.1 gives no statement for the case $n = 3$ and $w = 0$. In the next section we prove that in three dimensions the existence of a real Killing spinor of weight 0 is essentially equivalent to the fact that this manifold is a Gauduchon–Tod geometry:

Definition 1.2 (see [4], Proposition 5). A three-dimensional Weyl manifold (M^3, c, W) is called Gauduchon–Tod geometry, if there exists a density $\beta \in \Gamma(\mathcal{L}^{-1})$ such that the following conditions are satisfied:
1. W is Einstein–Weyl;
2. $R = 24\beta^2$;
3. $4\nabla\beta = *F$.

Remark. *The* $\kappa \in C^\infty(M, \mathbb{R})$ *in Proposition* 5 *of* [4] *is related to the* $\beta \in \Gamma(\mathcal{L}^{-1})$ *of Definition* 1.2 *in the following way:* $\kappa l_{g_\Sigma}^{-1} = -4\beta$. *For more information on Gauduchon–Tod geometries, e.g. their classification, see* [4] *and the references therein*.

Hence, the main result of this text is as follows:

Theorem 1.3. *Let* (M^3, c, W) *be a CSpin-manifold*.
1. *If* $\psi \in \Gamma(S^0)$ *is a real Killing spinor then the space of solutions of the Killing equation is two-dimensional and* (M^3, c, W) *is a Gauduchon–Tod geometry.*
2. *Conversely, any simply connected Gauduchon–Tod geometry has a two dimensional space of Killing spinors of weight 0*.

2. The proof of Theorem 1.3

Let (M^3, c, W) be a CSpin-manifold. The curvature tensor of the Weyl structure W is given by

$$\mathcal{R} = Ric^N \Delta c + F \otimes c,$$

where $\Delta : T^{2,0} \times T^{2,0} \to T^{4,0}$

$$\omega\Delta\eta := [(23) + (12)(24)(34) - (24) - (12)(23)]\omega\otimes\eta, \quad \omega, \eta \in T^{2,0}$$

is the so-called Kulkarni–Nomizu product (see [1]) and

$$Ric^N := -sym_0 Ric - \tfrac{1}{12}Rc + \tfrac{1}{2}F \tag{3}$$

is the normalized Ricci tensor of W (see [3]). sym_0 denotes the symmetric trace free part of a (2,0)-tensor. The following lemma is a tool for calculations with Kulkarni–Nomizu products in spin geometry:

Lemma 2.1. *Let ω be a (2, 0)-tensor. Then the following algebraic identity holds in any dimension:*

$$\mu^{34}\omega\Delta c = 2Alt\,\nu\mu^2\omega - 2Alt\,\omega.$$

Proof.

$$\begin{aligned}\mu^{34}\omega\Delta c &= \mu^{34}[(23)+(12)(24)(34)-(24)-(12)(23)]\omega\otimes c\\ &= [\mu^{24}+(12)\mu^{32}-\mu^{32}-(12)\mu^{24}]\omega\otimes c\\ &= [\mu^{23}+(12)\mu^{32}-\mu^{32}-(12)\mu^{23}]\omega\otimes c\\ &= [-2\mu^{32}+2tr^{23}+2(12)\mu^{32}-2(12)tr^{23}]\omega\otimes c\\ &= -2[\mu^{32}-(12)\mu^{32}]\omega\otimes c - 2Alt\,\omega\\ &= -2[1-(12)]\nu\mu^2\omega\otimes c - 2Alt\,\omega\\ &= 2Alt\,\nu\mu^2\omega - 2Alt\,\omega.\end{aligned}$$

□

Lemma 2.2. *Let $4\nabla\beta = *F$ be satisfied on (M^3, c, W). Then the following identities are true for any spinor $\psi \in \Gamma(S^w)$:*

$$\tfrac{1}{4}Alt\,\nu\mu^2 F\otimes\psi - Alt\,\nabla\beta\otimes\nu\psi - \tfrac{1}{2}F\otimes\psi = 0 \tag{4}$$

and

$$(\nu\nabla\beta - \nabla\beta\cdot\nu)\cdot\psi - \tfrac{1}{2}\mu^2 F\otimes\psi = 0. \tag{5}$$

Proof. Denote by (e_1, e_2, e_3) a local weightless conformal frame on (M^3, c) as well as $(\sigma_1, \sigma_2, \sigma_3)$ its dual. In dimension 3 we have the important relation

$$e_i\cdot e_j\cdot\psi = -\sum_{k=1}^{3}\epsilon_{ijk}e_k\cdot\psi. \tag{6}$$

Here ϵ_{ijk} denotes the Levi–Civita symbol. Since the $*$-operator on 2-forms is defined by the formula

$$*F = \frac{1}{2}\sum_{i,j=1}^{3}F(e_i, e_j)*(\sigma_i\wedge\sigma_j) = \frac{1}{2}\sum_{i,j,k=1}^{3}F(e_i, e_j)\epsilon_{ijk}\sigma_k$$

we can rewrite the assumption as follows:

$$8\nabla_{e_k}\beta = \sum_{i,j=1}^{3} \epsilon_{ijk} F(e_i, e_j). \tag{7}$$

We have

$$\begin{aligned} F \cdot \psi &= \sum_{i,j=1}^{3} F(e_i, e_j) e_i \cdot e_j \cdot \psi = -\sum_{i,j,k=1}^{3} \epsilon_{ijk} F(e_i, e_j) e_k \cdot \psi \\ &= -8 \sum_{k=1}^{3} (\nabla_{e_k}\beta) e_k \cdot \psi = -8\nabla\beta \cdot \psi. \end{aligned} \tag{8}$$

Using (6) and (7) we get

$$\begin{aligned} &\tfrac{1}{4} Alt\, \nu\mu^2 F \otimes \psi - Alt\, \nabla\beta \otimes \nu\psi - \tfrac{1}{2} F \otimes \psi \\ &\quad = \frac{1}{4} \sum_{i,j,k=1}^{3} F(e_j, e_k)\sigma_i \wedge \sigma_j \otimes e_i \cdot e_k \cdot \psi - \sum_{i,j=1}^{3} (\nabla_{e_i}\beta)\sigma_i \wedge \sigma_j \otimes e_j \cdot \psi \\ &\qquad - \frac{1}{4} \sum_{i,j=1}^{3} F(e_i, e_j)\sigma_i \wedge \sigma_j \otimes \psi \\ &\quad = -\frac{1}{4} \sum_{i,j=1}^{3} F(e_j, e_i)\sigma_i \wedge \sigma_j \otimes \psi + \frac{1}{4} \sum_{\substack{i,j,k=1 \\ i\neq k}}^{3} F(e_j, e_k)\sigma_i \wedge \sigma_j \otimes e_i \cdot e_k \cdot \psi \\ &\qquad - \sum_{i,j=1}^{3} (\nabla_{e_i}\beta)\sigma_i \wedge \sigma_j \otimes e_j \cdot \psi - \frac{1}{4} \sum_{i,j=1}^{3} F(e_i, e_j)\sigma_i \wedge \sigma_j \otimes \psi \\ &\quad = \frac{1}{4} \sum_{i,j,k=1}^{3} \epsilon_{kil} F(e_j, e_k)\sigma_i \wedge \sigma_j \otimes e_j \cdot \psi - \frac{1}{8} \sum_{i,j,k,l=1}^{3} \epsilon_{kli} F(e_k, e_l)\sigma_i \wedge \sigma_j \otimes e_j \cdot \psi \\ &\quad = \frac{1}{4} \sum_{i,j,k=1}^{3} \epsilon_{kij} F(e_j, e_k)\sigma_i \wedge \sigma_j \otimes e_j \cdot \psi \\ &\qquad - \frac{1}{8} \sum_{\substack{i,j,k,l=1 \\ j=k}}^{3} \epsilon_{jli} F(e_j, e_l)\sigma_i \wedge \sigma_j \otimes e_j \cdot \psi - \frac{1}{8} \sum_{\substack{i,j,k,l=1 \\ j=1}}^{3} \epsilon_{kji} F(e_k, e_j)\sigma_i \wedge \sigma_j \otimes e_j \cdot \psi \\ &\quad = \frac{1}{4} \sum_{i,j,k=1}^{3} \epsilon_{kij} F(e_j, e_k)\sigma_i \wedge \sigma_j \otimes e_j \cdot \psi \\ &\qquad - \frac{1}{8} \sum_{i,j,l=1}^{3} \epsilon_{lji} F(e_j, e_l)\sigma_i \wedge \sigma_j \otimes e_j \cdot \psi - \frac{1}{8} \sum_{i,j,k=1}^{3} \epsilon_{kij} F(e_j, e_k)\sigma_i \wedge \sigma_j \otimes e_j \cdot \psi = 0, \end{aligned}$$

i.e. we have shown (4). Contracting (4) by Clifford multiplication, using (8) and $\mu^2\nu = -\nu\mu^1 - 2Id$ yields

$$\begin{aligned}0 &= \tfrac{1}{4}\mu^2 Alt\,\nu\mu^2 F\otimes\psi - \mu^2\,Alt\,\nabla\beta\otimes\nu\psi - \tfrac{1}{2}\mu^2 F\otimes\psi\\ &= \tfrac{1}{4}\mu^2\nu\mu^2 F\otimes\psi - \mu^2\nabla\beta\otimes\nu\psi - \tfrac{1}{2}\mu^2 F\otimes\psi - \tfrac{1}{4}\mu^1\nu\mu^2 F\otimes\psi + \mu^1\nabla\beta\otimes\nu\psi\\ &= -\tfrac{1}{4}\nu F\cdot\psi - \tfrac{1}{2}\mu^2 F\otimes\psi + 3\nabla\beta\otimes\psi - \tfrac{1}{2}\mu^2 F\otimes\psi + \tfrac{3}{4}\mu^2 F\otimes\psi + \nabla\beta\cdot\nu\psi\\ &= -\tfrac{1}{4}\nu F\cdot\psi - \tfrac{1}{2}\mu^2 F\otimes\psi + 3\nabla\beta\otimes\psi - \tfrac{1}{2}\mu^2 F\otimes\psi + \tfrac{3}{4}\mu^2 F\otimes\psi + \nabla\beta\cdot\nu\psi\\ &= \nu\nabla\beta\cdot\psi + \nabla\beta\otimes\psi - \tfrac{1}{4}\mu^2 F\otimes\psi = \tfrac{1}{2}(\nu\nabla\beta - \nabla\beta\cdot\nu)\cdot\psi - \tfrac{1}{4}\mu^2 F\otimes\psi. \qquad \square\end{aligned}$$

Hence (5) is true.

Proof of Theorem 1.3. Let $\psi \in \Gamma(S^0)$ be a real Killing spinor. The first statement follows immediately from the existence of an equivariant quaternionic structure on the spinor module, which commutes with the Clifford multiplication. By Theorem 1.1 and its proof we have

$$R = 24\beta^2, \quad F\cdot\psi = -8\nabla\beta\cdot\psi.$$

Since ψ vanishes nowhere the second equation is equivalent to

$$4\nabla\beta = *F$$

by (6). Therefore we have already verified the conditions 2 and 3 of a Gauduchon–Tod geometry. It remains to be proved that the manifold is Einstein–Weyl. To this end we have to simplify (1) by means of $R = 24\beta^2$ and $Ric' = sym_0 Ric + \frac{1}{3}Rc - \frac{1}{2}F$ to

$$\mu^2 sym_0 Ric'\otimes\psi = -(\nabla\beta\cdot\nu - \nu\nabla\beta)\cdot\psi - \tfrac{1}{2}\mu^2 F\otimes\psi.$$

But the right-hand side vanishes according to (5). Hence W is Einstein–Weyl.

Conversely, let W be a simply connected Gauduchon–Tod geometry. It is sufficient to show that S^0 is flat with respect to $\nabla^\beta = \nabla^{S,0} - \beta\otimes\nu$. To this end, we have to prove that the curvature of ∇^β vanishes. We use the properties of Gauduchon–Tod geometries given in Definition 1.2, the result of Lemma 2.1 and the Eq. (4) and $\mathcal{R}^{S,0} = \frac{1}{4}\mu^{34}Ric^N\Delta c = -\frac{1}{4}\mu^{34}(\frac{1}{12}Rc - \frac{1}{2}F)\Delta c$.

$$\begin{aligned}\mathcal{R}^\beta &= Alt\,\nabla^\beta\circ\nabla^\beta = Alt\,\nabla^\beta\circ(\nabla^{S,0} - \beta\otimes\nu)\\ &= Alt(\nabla^{S,0}\circ\nabla^{S,0} - \nabla\beta\otimes\nu - (12)\beta\otimes\nu\nabla^{S,0} - \beta\nu\nabla^{S,0} + \beta^2\nu\nu)\\ &= \mathcal{R}^{S,0} - Alt(\nabla\beta)\nu + \beta^2\,Alt\,\nu^{12}\\ &= -\tfrac{1}{4}\mu^{34}(\tfrac{1}{12}Rc - \tfrac{1}{2}F)\Delta c - Alt(\nabla\beta)\nu + \beta^2 Alt\,\nu^{12}\\ &= -\tfrac{1}{4}(\tfrac{1}{6}R\,Alt\,\nu\mu^2 c - Alt\,\nu\mu^2 F + 2F) - Alt(\nabla\beta)\nu + \beta^2\,Alt\,\nu^{12}\\ &= -\tfrac{1}{24}R\,Alt\,\nu^{12} + \tfrac{1}{4}Alt\,\nu\mu^2 F - \tfrac{1}{2}F - Alt(\nabla\beta)\nu + \beta^2\,Alt\,\nu^{12}\\ &= \tfrac{1}{4}Alt\,\nu\mu^2 F - \tfrac{1}{2}F - Alt(\nabla\beta)\nu = 0. \qquad \square\end{aligned}$$

Acknowledgements

I would like to thank N. Ginoux for pointing out a gap in a former version of [2]. This concerns the real Killing equation in dimension 3 and for weight 0.

References

[1] A.L. Besse, Einstein Manifolds, Springer, Berlin, 1987.
[2] V. Buchholz, Spinor equations in Weyl geometry, Suppl. Rend. Civc. Mat. Di Palermo, math.DG/9901125 v2, in press.
[3] D. Calderbank, H. Pederson, Einstein–Weyl geometry, Odense Universitet, preprint, No. 40, 1997.
[4] P. Gauduchon, K.P. Tod, Hyper-Hermitian metrics with symmetry, J. Geom. Phys. 25 (1998) 291–304.